ON MEAN-FIELD GAMES

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partially supported by "Chair FDD" (IEF: EDF + CALYON)

SUMMARY

I INTRODUCTION

• GOAL : Introduce, Justify, Analyse and Apply

a new class of mathematical models

for the study of the "average" behavior of a very large number of "rational" agents in interaction

• Mean-Field, Self-Consistent type models

> Nash equilibria for "continua of players"

(# game-theory : Aumann, Carmona ...)

• Rat. (economics): "Optimal behavior"

• New mathematical problems > many classical problems

(kinetic models, Fluid Mechanics, Hartree type eqs., semilinear eqs...)
+ HJB - stochastic control

• Simplification of games and equilibria in the continuum limit

II A simple ex. in Optimization

states $x_1, \dots, x_N \in \mathbb{R}^d$

player i : criterion $F_i(X)$ $\quad X = (x_1, \dots, x_N)$

Nash point: $(\bar{x}_1, \dots, \bar{x}_N)$ s.t.

$$\bar{x}_i \text{ min. } F_i(\bar{x}_1, \dots, \bar{x}_{i-1}, \cdot, \bar{x}_{i+1}, \dots, \bar{x}_N)$$

(ex.: vacations !)

Natural but plagued with diff.: existence?, uniqueness?, stability?, num. comp.? ...

? $N \to \infty$: "identical" players" ?

$$F_i(X) = F(x_i; \underbrace{(x_j)_{j \neq i}}_{\text{symmetric}}) \quad (\text{or } \underset{N}{\to} F \dots)$$

$$F \in \text{"} C(\mathbb{R}^d, P(\mathbb{R}^d)) \text{"} \quad F(x_i; \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j})$$

ex.: $F_i = F_0(x_i) + f\left(\frac{1}{N|B_\varepsilon|} \# \{j \neq i \,/\, |x_i - x_j| < \varepsilon\}\right) \quad (\varepsilon > 0)$

$F(x, m) = F_0(x) + f(m * 1_{B_\varepsilon} / |B_\varepsilon|) \quad \left(\begin{array}{l} f \uparrow \text{ aversion} \\ f \downarrow \text{ desire} \end{array} \right)$

. Requires new mathematical tools → other applications

PDE's and dim) → ∞
large deviations for SPDE
theor. φ
statistical φ and particles in interaction...
? num. analysis : flows of grids... ?

. Particle (matter element) = choiceless agent!

• Models for economics and finance, but...

. Applications to Economics and Finance → J.M. Lasry

- Price formation
- volatility formation
- bubbles (fund managers)
- dates

J.M. Lasry, PL[2] + O. Guéant : distribution of wealth (Pareto's tails)
community islands

with P.N. Giraud (ENSMP) : Hotelling's rent and Hubbert peak for oil prices

techn. "breakthrough" - thermal isolation of houses
→ model for EDF

Refs: 2 notes Comptes Rendus, 1 survey paper (Jap. J. Math. Soc..)
web site : CdF courses 06-07, 07-08, 08-09

"THM" : ALL NASH POINTS CV AS $N \to \infty$ TO A SOL. OF (MFG) MODEL (up to subsequences)

m N. distribution of players $\in \mathcal{P}(\mathbb{R}^d)$
$\forall x \in \mathrm{Supp}\, m$ generic player

(MFG) x is a min of $F(\cdot\,; m)$

(MFG) $m \in \mathcal{P}(\mathbb{R}^d)$, $\mathrm{Supp}(m) \subset \mathrm{ArgMin}\, F(\cdot\,; m)$

$\iff$ m min over $\mathcal{P}(\mathbb{R}^d) \ni f$: $\int F(x;m)\, f(x)$

Facts : - proof : elementary (but not so simple)

- existence, stability of some equil., num. comp.
- uniqueness if F mon. on $\mathcal{P}$

$$\int (F(x;f_1) - F(x;f_2))(f_1 - f_2) \geq 0$$

- efficiency : if $\exists$ potential Φ (over $\mathcal{P}$) $\Phi'(m) = F(x;m)$

MFG $\underset{\text{if convex}}{\rightleftarrows}$ $\underset{\mathcal{P}}{\mathrm{Min}}\, \Phi(m)$

ex. $(\varepsilon = 0)$ $F(x;m) = F_0(x) + f(m(x))$

uniq. (efficiency) if $f \uparrow$ (aversion)

nonuniq. if $f \downarrow$ (desire)

III The Math. framework for $N \to \infty$

Functions of N var. as $N \to \infty$?

Differential info., PDE's as $N \to \infty$?

Answer : IF SYMMETRY, THEN WORK ON $\mathcal{P}$!

(with Monge-Kantorovich-Wasserstein metrics d_1, d_2, d_∞)

$$(x_1, \dots, x_N) \in \underbrace{Q^N/S_N}_{\text{permutation}} \longleftrightarrow \frac{1}{N}\sum_i \delta_{x_i} \in \mathcal{P}(Q)$$

(Q = periodic box = torus, $\mathbb{R}^d$, metric space ....)

$\Rightarrow$ previous partial observ.: Grunbaum (B.), Malytsin (T.-P.)

THM 1 : $\begin{cases} u_N \in C(Q^N) \text{ sym. in } (x_1, \dots, x_N) \\ \text{"uniform modulus"} \end{cases}$

then (up to subsequences) $\exists\, U \in C(\mathcal{P})$

$$\sup_{X \in Q^N} \left| u_N(X) - U\Big(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}\Big) \right| \xrightarrow[N]{} 0$$

yields a simple proof of H.S. thm, + duality with H.S., extensions (moduli, several groups, reiteration ...), p.sym., approximation ...

About the W. metrics (cf Otto, Brenier, Villani, Ambrosio ...
(↪ transport, PDE's, geometry, ...)

ex. $$d_{(2)}(m_1, m_2)^2 = \operatorname{Inf}_{(X,Y)} \{ E[|X-Y|^2] \,/\, \text{law of } X = m_1, \text{ law of } Y = m_2 \}$$

$$d(\delta_x, \delta_y) = |x-y|$$

$$d\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{y_i}\right)^2 = \inf_{\sigma \in S^N} \frac{1}{N} |X - Y_\sigma|^2$$

$Q^N/S^N \hookrightarrow \underline{P}(Q)$ (compactness: foldings but singularities ...)

Important equivalent formulation

$U \in C(\underline{P})$ (d metric)

$\Updownarrow$

$U \in C(L^2)$ $L^2 = L^2(\Omega)$ (Ω nontrivial proba. space)

AND $U(X) = U(Y)$ if $X \underset{d}{\sim} Y$

Allows to use both compactness ($\underline{P}$) and smoothness (L^2)

$\longrightarrow$ (new) diff. calc. on $\underline{P}$ (u_N "smooth" $\to$ U "smooth")

and PDE's on $\underline{P}$ (limits of PDE's in $(\mathbb{R}^d)^N$ ...)

IV A CLASS OF DYN. EXS

STOCHASTIC CONTROL FOR EACH AGENT, IND. NOISES...

- FORMAL DERIVATION $\longrightarrow$ MFG
- JUSTIFICATION : NASH EQUIL. $N<\infty$ (cf for ex. Bensoussen - Frehse), $N \to \infty$ PDE on $Q \times \mathcal{P}(Q)$, partial integr. along characteristics in $\mathcal{P}$ $\longrightarrow$ MFG

ex. $T<\infty$

$$
(MFG) \begin{cases} \dfrac{\partial u}{\partial t} + F(D_x^2 u, D_x u, u, x, t; m) = 0 \\ \dfrac{\partial m}{\partial t} - D_x^2 \cdot (F_A m) - D_x \cdot (F_p m) = 0 \\ m|_{t=0} = m_0 \in \mathcal{P} \quad , \quad u|_{t=T} = U(x; m(T)) \end{cases}
$$

$F = F(A, p, \ldots)$ convex in (A, p) (stoch. control) for ex.
$\nearrow$ in A

interpretation: u is the value f^{ion} (opt. criterion) of a generic agent $\longrightarrow$ HJB eq. with dep. upon m the density of players, yields an optimal strategy along which evolves m

Backwd HJB eq $\longleftrightarrow$ Fwd FP eq.

limit terminal condition : $m(T) = m_1$ (planification)

EXAMPLES AND PARTICULAR CASES

- non m dep.: HJB eq.
- no u dep.: FP eq, Porous Media, Vlasov (Boltzmann eq.)
- A class of exs: $\nu \geq 0$, $H(x,p)$ convex in p

$$\frac{\partial u}{\partial t} + \frac{\nu}{2}\Delta u + H(x, \nabla u) = F[m]$$

$$\frac{\partial m}{\partial t} - \frac{\nu}{2}\Delta m - \operatorname{div}_x\left(\frac{\partial H}{\partial p}(x, \nabla u)\, m\right) = 0$$

- $\nu = 0$, $H = \frac{1}{2}|p|^2$, $F[m](x) = F(m(x))$

  Compressible Euler Eqs (barotropic)

- $\nu = 0$, $H = \frac{1}{2}|p|^m$, $F \equiv 0$, $m(T) = m_1$

  Optimal Transport and Wasserstein distances

- stationary problems, $H = \frac{1}{2}|p|^2 - V_0(x)$, $\nu = 1$

$$\Longleftrightarrow \begin{cases} -\Delta\psi + \left(V_0 + F[\psi^2]\right)\psi = (\lambda\psi) \\ \int \psi^2 = 1 \end{cases}$$

$\begin{cases} \text{all semilinear elliptic equations} & F[m](x) = F(m(x)) \\ \text{Hartree equations} & F[m] = V_1 * m \end{cases}$

- several populations: population dynamics for $m_1, m_2, \dots$
  or birth and death

...

Mathematical results and perspectives

- existence: strong, weak, possibly nonexistence for local F ...
- propagation $N \to \infty$
- uniqueness: T small or F monotone, U monotone
  (to be compared with the known results in the eqns ...)
- stationary problems
- Fwd/Bwd ODE's on $[0, +\infty]$
- Extensions to several populations, free bdry pbs
- Hamiltonian structure if $F = \Phi'(m)$ / optimal control
  And Optimal control of PDE's (FP $\underset{or}{\Longleftrightarrow}$ HJB).
- numerical analysis / numerical computations
- randomly heterogeneous populations
- Applications
- Correlations

⋮

- Partial observations
- recursive utility